Planar graphs are measure treeable

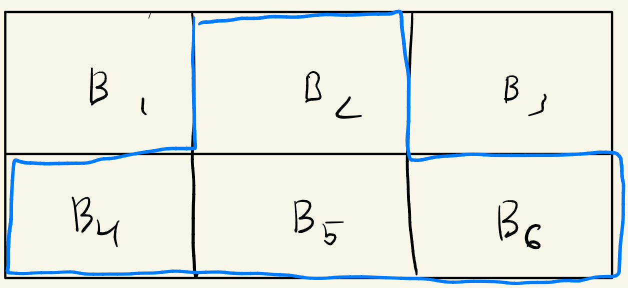
Recall

2-basis of a graph is a collection $\mathcal{B}$ of simple cycles s.th

i) no edge e belongs to more than two elts of $\mathcal{B}$

ii) $\mathcal{B}$ generates all cycles ie if C is a cycle in G, then there are $B_1, \dots, B_n \in \mathcal{B}$

$$\mathbb{1}_C = \sum_{i=1}^{n} \mathbb{1}_{B_i} \pmod 2$$



An <u>accumulation-free planar embedding</u> of a graph is a planar embedding s.th every compact subset of $\mathbb{R}^2$ intersects finitely many vertices and edges

A *face* of a planar embedding is a ~~bdd~~ connected component of the complement of the embedding

If F is a face ∂F is either a cycle or a bi-infinite line

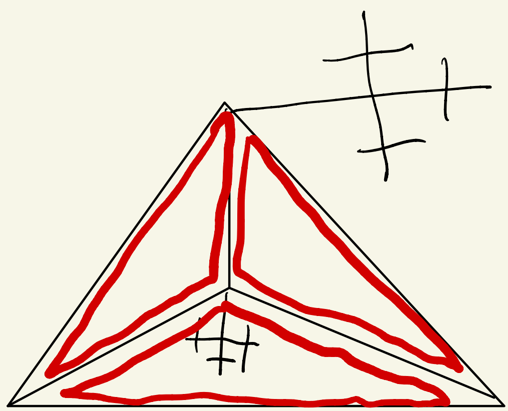
If it is a cycle, then we call it a *facial cycle*.

Thm (Thomassen) G 2-connected locally finite graph

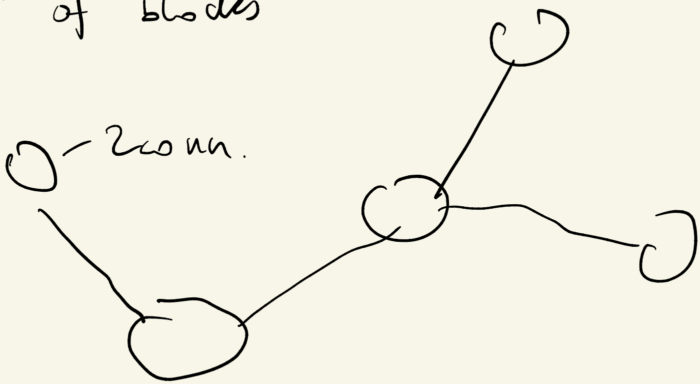
1) If G admits an accumulation-free planar embedding, then $\mathcal{B}$ the set of all facial cycles is a 2-basis of G

2) If G has a 2-basis $\mathcal{B}$, then there exists an acc. free planar embedding s.t. $\mathcal{B}$ is the set of all facial cycles.

Remark 1)



2) any graph has canonical <u>blocks</u> which are maximal 2-connected components and the graph is a "tree" of blocks

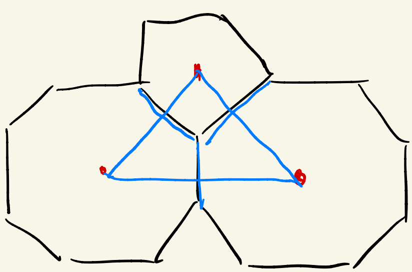


The dual of graph G with a 2-basis $\mathcal{B}$

G^* whose vertices are the elements of $\mathcal{B}$

if e is and edge in G s.th e belong to two elts of $\mathcal{B}$, say B_1, B_2

e^* is edge between B_1, B_2



Proposition G locally finite 2-connected graph which has a 2-basis $\mathcal{B}$. Assume that every edge belongs to two elements of $\mathcal{B}$.

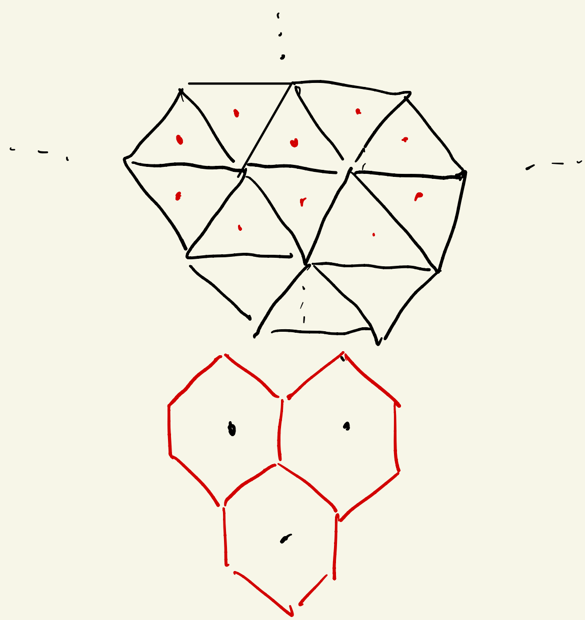
Then G^* has a 2-basis: if v is a vertex in G

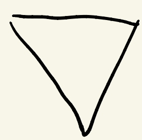
$$v^* = \{ e^* : v \in e \} \text{ is a simple cycle in } G^*$$

$$\mathcal{B}^* = \{ v^* : v \in G \} \text{ is a 2-basis of } G^*$$

$$G \simeq G^{**}$$

$$v \longmapsto v^*$$

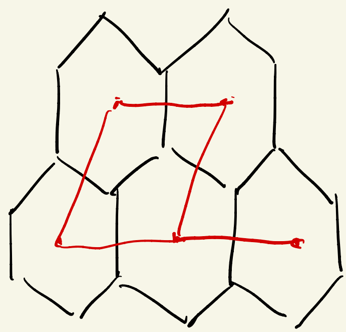




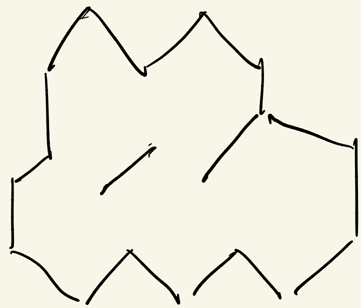
Def Suppose G is a planar graph with a 2-basis $\mathcal{B}$, G^* - its dual, $G_0^* \subseteq G^*$ is a subgraph

$$G \ominus G_0^* = G \setminus \{e \in G : e^* \in G_0^*\}$$

Ex



$G \ominus G_0^*$



Thm G locally finite aperiodic, 2-connected graph with a 2-basis $\mathcal{B}$.

Assume every edge of G belongs to two elements of $\mathcal{B}$.

G^* - dual. $G_0^* \subseteq G^*$ spanning subgraph

$$H = G \ominus G_0^*$$

1) H is acyclic iff G_0^* is aperiodic

2) G_0^* is acyclic iff H is aperiodic

3) H is a spanning tree (with the same components as G)

iff

G_0^* is a one-ended subtree.

Pf Assume G, G^* are connected

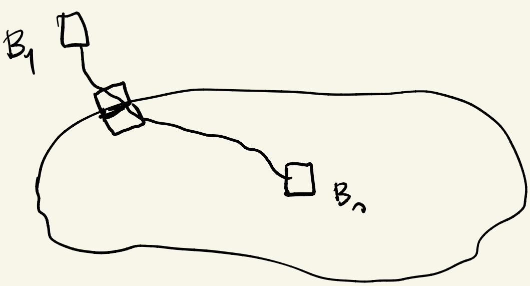
1) $\Rightarrow$

suppose G_0^* is aperiodic

Let C be a cycle in G

we want to show that C does not survive to H

we draw C on the plane and see it as a simple curve in $\mathbb{R}^2$



By Jordan, it divides $\mathbb{R}^2$ into two conn. components:

one bounded, one unbounded.

Let B_0 be a facial cycle in the bounded component

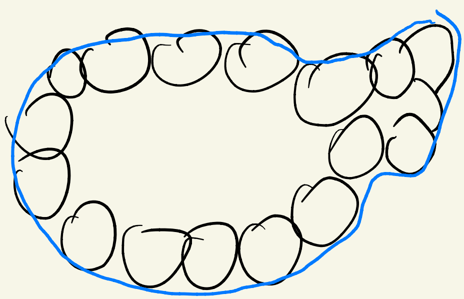
There are fin. many facial cycles in the bdd connected comp., so there is a facial cycle B_1 in the unbounded comp. s.t. B_0, B_1 are connected in G_0^*

the edge cut by a path from B_0 to B_1 does not survive to H.

$\Leftarrow$ suppose G_0^* has a finite component C

we will find a cycle in H

F



Let $K = \cup C$

Let F be an unbounded face of K

∂F - is the cycle

∂F is contained in H because every edge on ∂F touches only one facial cycle of G_0^*

2) follows from 1) by duality

3)

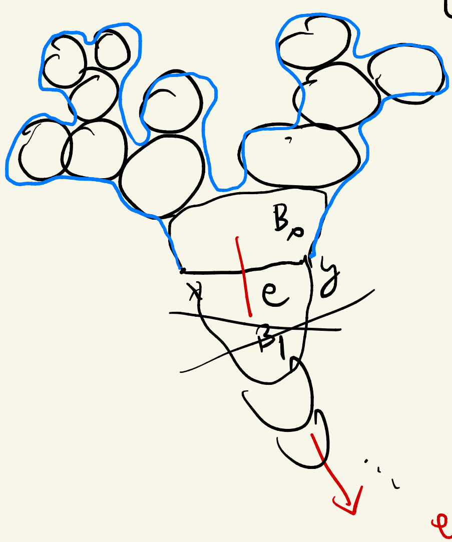
$\Leftarrow$ assume G_0^* is a one-ended subtree

WTS H is a spanning tree (connected)

By 1) H is acyclic, so we need to show H is connected

let $x, y \in H$ assume there is an edge e between them in G.

WTS that there is a path in H from x to y.



let B_0, B_1 be faces / cycles adjacent to e,

B_1 closer to the end

Removing B_1 leaves B_0 in a finite component of G_0^*

call it D

Let $P = \partial D \setminus \{e\}$ (the blue path)

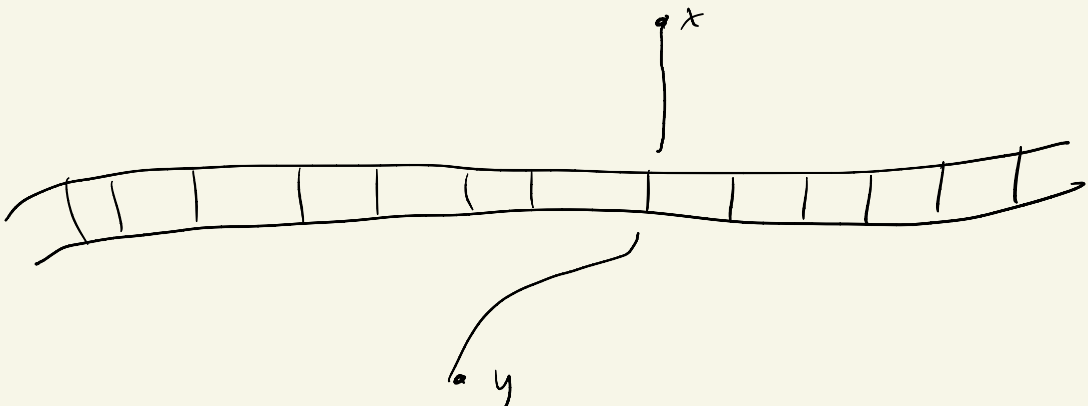
every element of P survives in H,
so x and y are connected in H.

$\Rightarrow$ assume H is connected tree

WTS G_0^* is a one-ended subtree

By 2) G_0^* is a subtree, all that is left to see is that G_0^* is one-ended

If G_0^* is not one-ended, then H (it is aperiodic), it has a biinfinite path



If x, y are in different components of its complement, then x, y are not connected in H. $\square$

Proposition A

G - loc finite Borel graph on X

$A \subseteq X$ Borel

$\exists f : [A]_G \setminus A \to [A]_G$ acyclic and 1-ended, i.e.

$\forall x \quad \bigcup_{n=1}^{\infty} f^{-n}(x)$ is finite

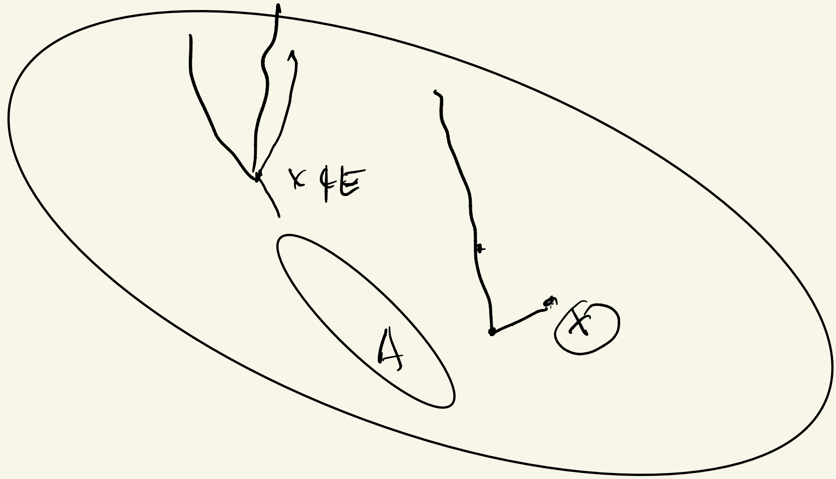
$f \subseteq G \qquad f$ - Borel.

Pf WLOG $[A]_G = X$

$E = \{x : \exists x_0 = x, x_1, x_2, \ldots$
$d(x_{n+1}, A) > d(x_n, A)\}$

E - Borel

$$f(x) = \begin{cases} y \in E & d(x,A) < d(y,A) \quad x \in E \\ y & d(y,A) < d(x,A) \quad x \notin E \end{cases}$$



$\square$

Proposition B Let $\underline{G}$ (G^*) loc. finite Borel a.n. 2-ended

if $\exists\, T \subseteq G$ acyclic with $E_T = E_G$
$\Rightarrow G$ admits a 1-ended spanning forest.

Side remark This follows from

Proposition B' Let $\underline{T}$ acyclic aperiodic loc finite Borel

If $\mathcal{T}$ a.n. 2-ended
$\Rightarrow$ T contains a 1-ended sp. subfor.

Def E an ctble Borel equel (CBER)
μ-μamenable
if $\exists \varphi_n : E \to [0,\infty)$

1) $\sum_{y \in [x]_E} \varphi(x,y) = 1 \quad \forall^{\mu} x$

2) $\forall x, x' \quad xEx'$

$$\lim_{n\to\infty} \sum_{y\in[x]_E} |\varphi_n(x,y) - \varphi_n(x',y)| \to 0$$

Thm (Connes-Feldman-Weiss)
μ-hyperfiniteness $=$ μ-amenability

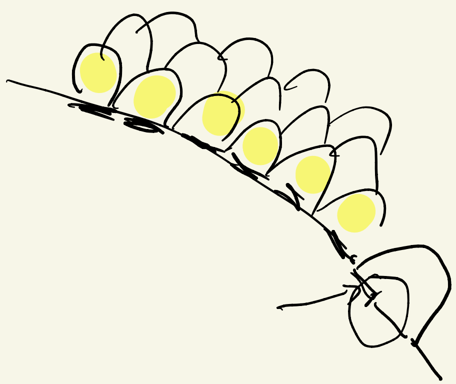
Thm let G be loc. finite Borel planar
Then $\exists$ an acyclic subgraph $T \subseteq G$
$E_T = E_G$.

Pf WLOG every comp. of G is 2-conn.

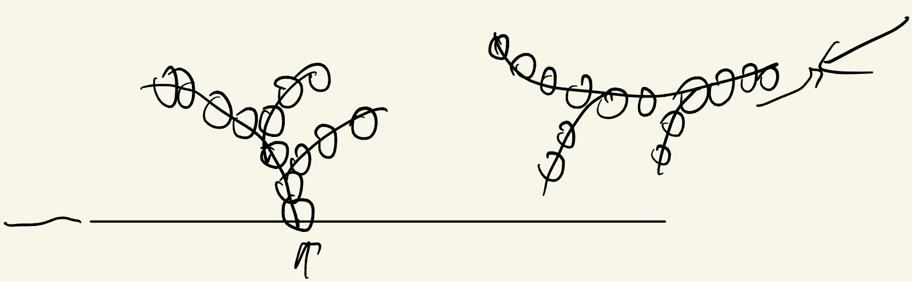
Step 1 G_1 set of edges which belong to 1 facial cycle

B_1 - set of facial cycles that touch G_1

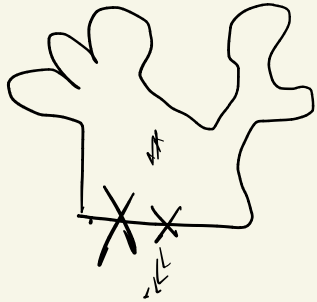
Apply Prop A to produce a 1-ended f flow ~~G~~ and G_0^*



$A = B_1$ and the dual ~~of~~ G^* on those comp. where $G_1 \neq \emptyset$



$T = \underline{G \ominus G_0^*}$ "some edges on the boundary"



From now on, assume $G_1 = \emptyset$

$\Rightarrow$ G, G^* are 1-ended

faces cover the whole plane

Let $T \subseteq G$ be acyclic aperi.

<u>hyperfinite</u>

T^*

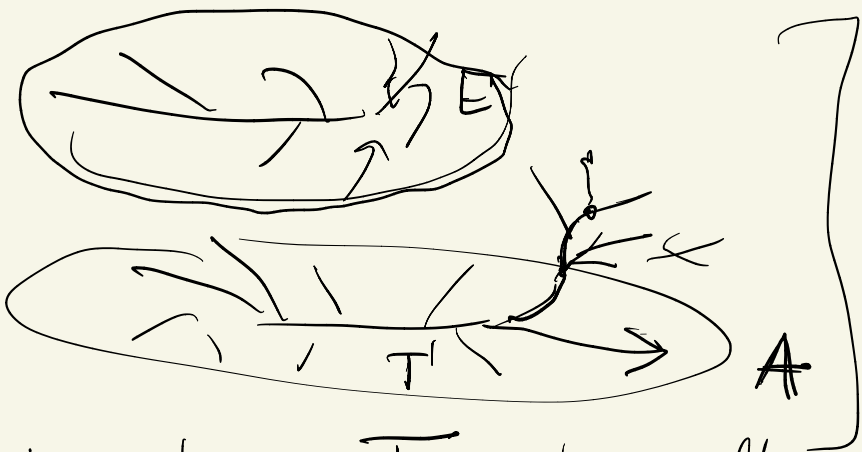
<u>Step 2</u> $\underline{Y}$ – G-saturation of

the set of comps of T that do not have 2 ends

let $T' \subseteq T$: {2-ended comps} $\cup$ 1-ended

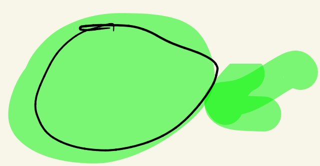
let A be the union of $\neq 2$ ended comps of T in Y

By Prop A applied to A



we now have T_0 whose all comps are 1-ended.

$$L = G^* \odot T_0$$



L is a subtree

We use prop B to G^* with

the subtreeing L

Then there exists a 1-ended spanning subforest of G^* and we are done by last time.

From now on assume all comp. of $\underline{T \text{ are } 2\text{-ended}}$

(or all comp of T^* are 2ended)

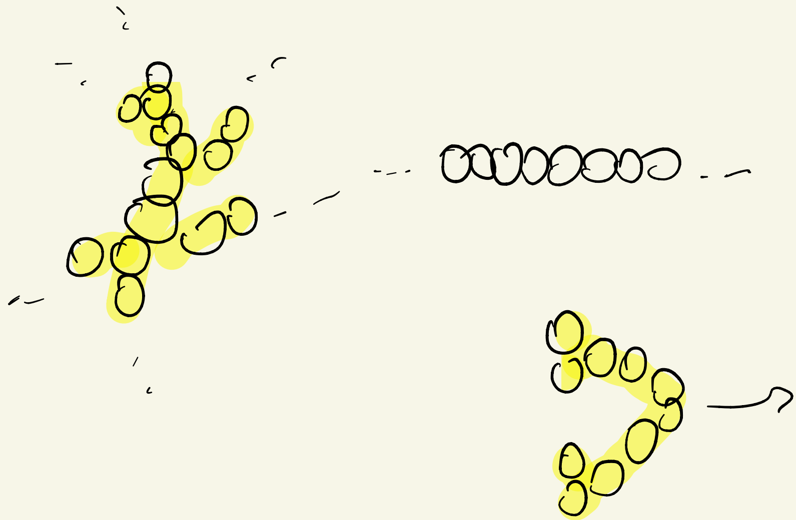
<u>Step 3</u>

$$H^* = G^* \ominus T$$

H^* is acyclic aperiodic subgraph of G^*

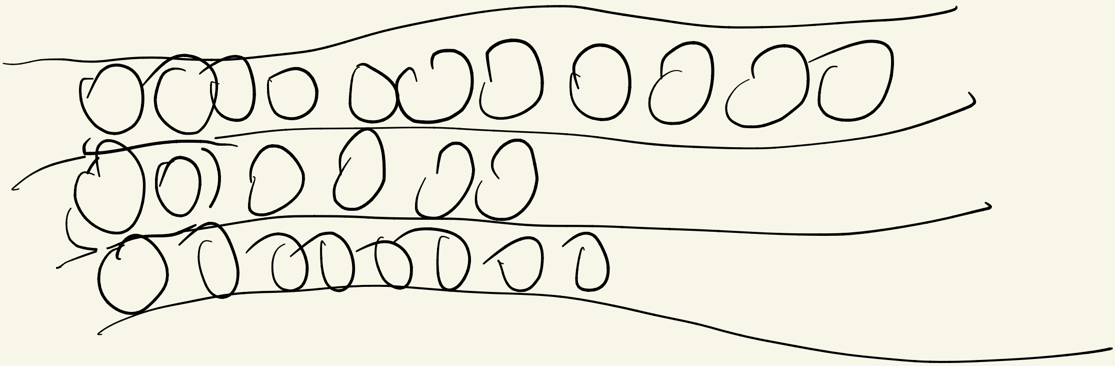
Z^* - saturation of $\neq 2$ ended comp of H^*

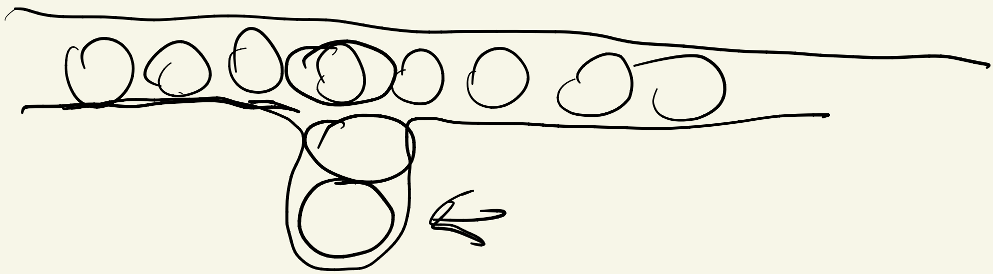
$Z = \partial Z^*$



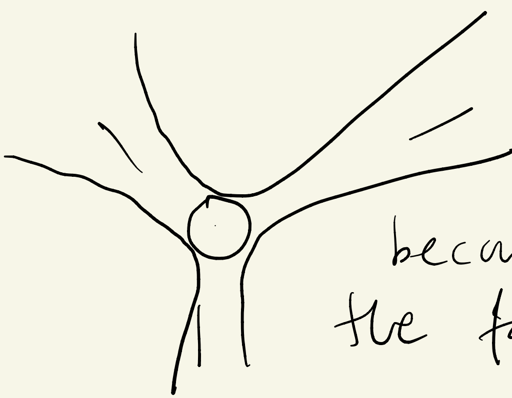
By Prop B' applied to $H^* \mid Z^*$ we can get a 1-ended span subforest of $H^* \mid Z^4$, which 1-ended subforest of G^*.

From now on we restrict attention to $X \setminus Z$ and hence assume all H^*-comp. are 2 ended.



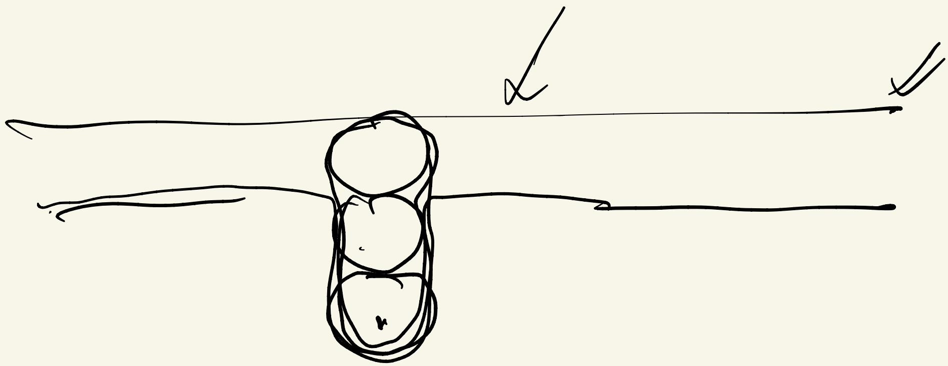


Note that every face in $X \setminus Z$ can intersect ≤ 2 lines from T

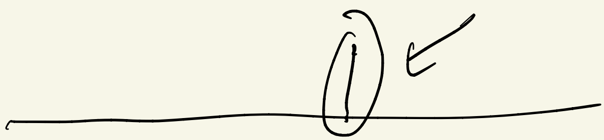


because otherwise the face is in Z

Step 4 We "make sure" that every face intersects 2 lines from T

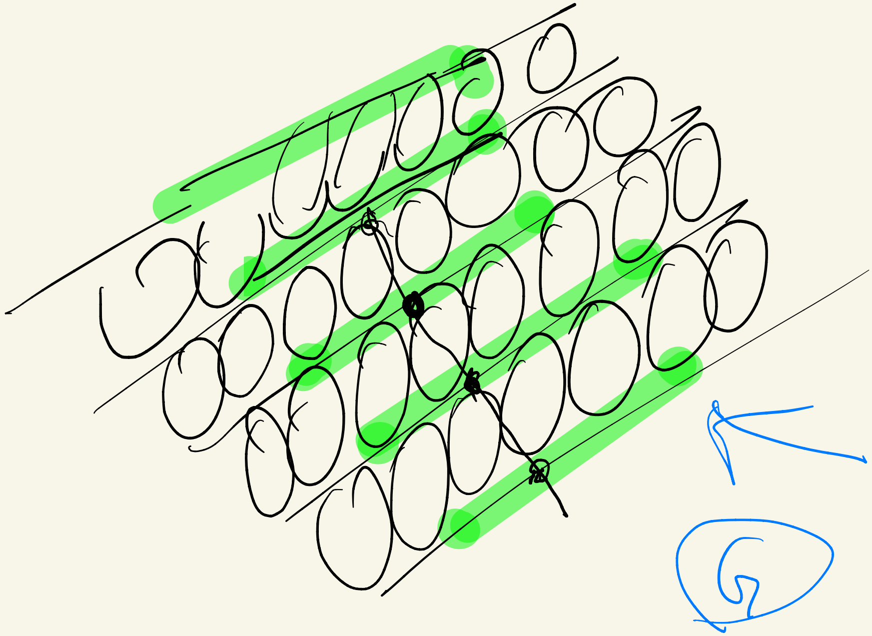


we merge cells together to get the structure we want

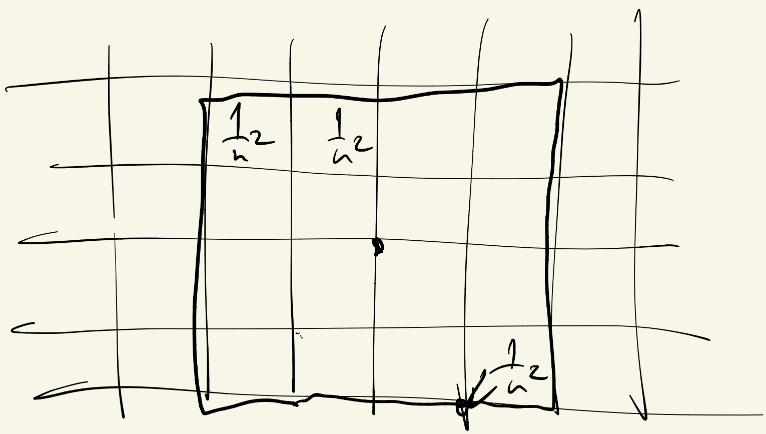


Step 5

we have finitely lines of T squeezed in between basic finite lines of H^*



We show that G is hyperfinite by showing H is amenable the same way $\mathbb{Z}^2$ is amenable



φ_n

$\square$

Cor $\pi_1(S)$ is stretchable

for any surface S.